4 Spectral LES for isotropic turbulence

We have seen that a major drawback of the eddy viscosity assumption in physical space is the nonexistence of a spectral gap between resolved and subgrid scales. This is an argument in favor of working in Fourier space, where we will see that the lack of a spectral gap may be dealt with in some sense.

4.1 Spectral eddy viscosity and diffusivity

We assume that the Navier–Stokes equation is written in Fourier space. This requires statistical homogeneity in the three directions of space, but we will see in the following how to handle flows with only one direction of inhomogeneity. Let $\hat{u}_i(\vec{k}, t)$ and $\hat{\rho}(\vec{k}, t)$ be the spatial Fourier transforms of, respectively, the velocity and passive-scalar fields introduced in Chapter 1. As already stressed, they are defined in the framework of generalized functions.[1] The filter consists of a sharp cutoff filter simply clipping all the modes larger than k_C, where $k_C = \pi/\Delta x$ is the cutoff wavenumber obtained when one uses a pseudo-spectral method in a given direction of periodicity.

We write the Navier–Stokes equation in Fourier space as

$$\begin{aligned}\frac{\partial}{\partial t}\hat{u}_i(\vec{k}, t) &+ [\nu + \nu_t(\vec{k}|k_C)]k^2\hat{u}_i(\vec{k}, t)\\ &= -ik_m P_{ij}(\vec{k})\int_{|\vec{p}|,|\vec{q}|<k_C}^{\vec{p}+\vec{q}=\vec{k}} \hat{u}_j(\vec{p}, t)\hat{u}_m(\vec{q}, t)d\vec{p}. \qquad (4.1)\end{aligned}$$

The spectral eddy viscosity $\nu_t(\vec{k}|k_C)$ is defined by

$$\nu_t(\vec{k}|k_C)k^2\hat{u}_i(\vec{k}, t) = ik_m P_{ij}(\vec{k})\int_{|\vec{p}|\text{or}|\vec{q}|>k_C}^{\vec{p}+\vec{q}=\vec{k}} \hat{u}_j(\vec{p}, t)\hat{u}_m(\vec{q}, t)d\vec{p}. \qquad (4.2)$$

[1] Discretized equivalents correspond to the discrete Fourier transforms of flows in spatially periodic domains.

At this point, it may not be positive or even real. The condition $\vec{p}+\vec{q}=\vec{k}$ is a "resonant-triad condition" resulting from the convolution coming from the Fourier transform of a product. The r.h.s. of Eq. (4.1) corresponds to a resolved transfer. A spectral eddy diffusivity for the passive scalar may be defined in the same way by writing the passive-scalar equation in Fourier space

$$\frac{\partial}{\partial t}\hat{\rho}(\vec{k},t)+[\kappa+\kappa_{\mathrm{t}}(\vec{k}|k_{\mathrm{C}})]k^2\hat{\rho}(\vec{k},t)=-ik_j\int_{|\vec{p}|,|\vec{q}|<k_{\mathrm{C}}}^{\vec{p}+\vec{q}=\vec{k}}\hat{u}_j(\vec{p},t)\hat{\rho}(\vec{q},t)d\vec{p} \tag{4.3}$$

with

$$\kappa_{\mathrm{t}}(\vec{k}|k_{\mathrm{C}})k^2\hat{\rho}(\vec{k},t)=ik_j\int_{|\vec{p}|_{\mathrm{or}}|\vec{q}|>k_{\mathrm{C}}}^{\vec{p}+\vec{q}=\vec{k}}\hat{u}_j(\vec{p},t)\hat{\rho}(\vec{q},t)d\vec{p}. \tag{4.4}$$

Expressions (4.2) and (4.4) give exact expressions of the eddy coefficients. They are, however, useless because they involve subgrid quantities. In fact, the eddy coefficients can be evaluated at the level of kinetic-energy and passive-scalar spectra evolution equations obtained with the aid of two-point closures of three-dimensional isotropic turbulence.

It is in this context that the concept of k-dependent eddy viscosity was first introduced by Kraichnan [147]. The spectral eddy diffusivity for a passive scalar was introduced by Chollet and Lesieur [42]. Kraichnan used the so-called test-field model. We work using a slightly different closure called the eddy-damped quasi-normal Markovian theory introduced by Orszag [224, 225] (see also André and Lesieur [6] and Lesieur [170] for details). We first briefly recall the main lines of this model.

4.2 EDQNM theory

In the EDQNM theory, which is easily manageable only in the case of isotropic turbulence, the fourth-order cumulants in the hierarchy of moments equations are supposed to relax the third-order moments linearly in the same qualitative way that the molecular viscosity does. Thus, a time θ_{kpq} characterizing this relaxation is introduced. The EDQNM gives for isotropic turbulence the following evolution equation for the kinetic-energy spectrum $E(k,t)$:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+2\nu k^2\right)&E(k,t)\\ &=\iint_{\Delta_k}dp\,dq\,\theta_{kpq}(t)\frac{k}{pq}b(k,p,q)E(q,t)[k^2E(p,t)-p^2E(k,t)],\end{aligned} \tag{4.5}$$

where the integration is carried out in the domain Δ_k of the (p, q) plane such that (k, p, q) can be the sides of a triangle and thus satisfy triangular inequalities. The nondimensional coefficient

$$b(k, p, q) = \frac{p}{k}(xy + z^3) \tag{4.6}$$

is defined in terms of the cosines (x, y, z) of the interior angles of the triangle formed by the resonant triad $(\vec{k}, \vec{p}, \vec{q})$. The time $\theta_{kpq}(t)$ is given by

$$\theta_{kpq} = \frac{1 - e^{-[\mu_{kpq}+\nu(k^2+p^2+q^2)]t}}{\mu_{kpq} + \nu(k^2 + p^2 + q^2)} \tag{4.7}$$

with

$$\mu_{kpq} = \mu_k + \mu_p + \mu_q$$

and

$$\mu_k = a_1 \left[\int_0^k p^2 E(p, t) dp\right]^{1/2}. \tag{4.8}$$

The constant a_1 is adjusted in such a way that the kinetic-energy flux is equal to ϵ in a Kolmogorov cascade of infinite length, as done in André and Lesieur [6]. One finds $a_1 = 0.218\ C_K^{3/2}$. An analogous equation may be written for the passive-scalar spectrum $E_\rho(k, t)$ with a scalar transfer involving products EE_ρ. Let us present now some recent EDQNM results of decaying isotropic turbulence at high or very high Reynolds number obtained by Lesieur and Ossia [174]. The code used is the one developed by Lesieur and Schertzer [164] in which nonlocal interactions[2] are treated separately and included analytically in the kinetic-energy transfer term in the EDQNM spectral evolution equation. Details are also given in Lesieur ([170], pp. 231–235). Wavenumbers are discretized logarithmically in the form

$$k_L = \delta k\ 2^{(L-1)/F}, \tag{4.9}$$

with L ranging from 1 to a maximum value L_S. In all calculations, F was taken equal to 8, which is twice as large as used in former calculations of this type done in Grenoble and should guarantee a higher precision.[3] Calling $k_{\max}$ the maximum wavenumber, we have also

$$\frac{k_{\max}}{k_i(0)} = A R_{k_i(0)}^{3/4}, \tag{4.10}$$

[2] Nonlocal interactions are those involving extremely distinct wavenumbers and thus very elongated triads.

[3] Comparisons with calculations done with $F = 4$ show that the difference of results is not very substantial, and so the latter value should be recommended, considering the much shorter computational times in this case.

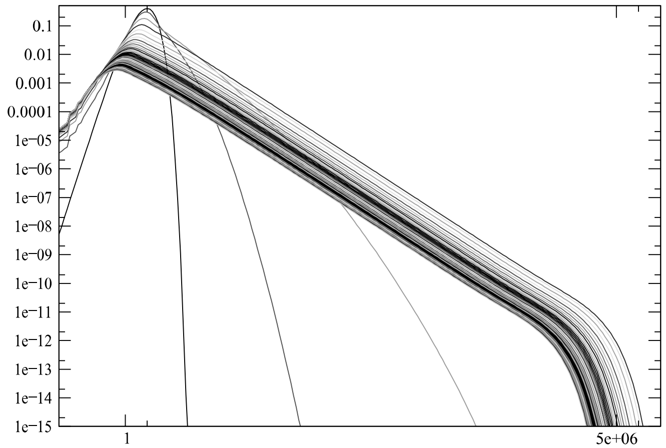

Figure 4.1. Kinetic-energy spectrum evolution in a decaying EDQNM calculation with $R_{k_i(0)} \approx 1.70 \times 10^9$.

with A equal to 1 and 3 in the decaying and forced calculations, respectively ($R_{k_i(0)}$ is a large-scale Reynolds number defined momentarily). This is lower than the value 8 proposed in Lesieur [170], but it permits a good-enough capturing of the dissipative range and results in a substantial reduction of computing time. These calculations have in fact been done on a PC/LINUX machine.

In decaying calculations, the initial kinetic-energy spectrum is

$$E(k,0) = A_s\, k^s \, \exp\left[-\frac{s}{2}\frac{k^2}{k_i(0)^2}\right], \tag{4.11}$$

where A_s is a normalization constant chosen such that $\int_0^{k_{\max}} E(k,0)dk = \frac{1}{2}v_0^2 = \frac{1}{2}$. The time unit is the initial large-eddy turnover time $[v_0 k_i(0)]^{-1}$. The constant a_1 corresponds to $C_K = 1.40$. The initial large-scale Reynolds number is $R_{k_i(0)} = v_0/\nu k_i(0)$.

We first present a calculation with $s = 8$, $\delta k = 0.125$, $k_i(0) = 2$, and $R_{k_i(0)} \approx 1.70 \times 10^9$. Figure 4.1 displays the time evolution of the kinetic-energy spectrum $E(k,t)$ for this run, up to 100 turnover times. We see very clearly the establishment of an ultraviolet inertial-type range whose slope may be checked to be (on this log–log plot) very close to the $k^{-5/3}$ Kolmogorov law along more than five decades. In fact this point will be explored later by considering compensated spectra $\epsilon^{-2/3}k^{5/3}E(k,t)$. We see also on the figure the rapid formation of a k^4 infrared spectrum. This corresponds to the k^4 infrared spectral backscatter, which will be discussed later. At the end of the evolution ($t = 100$), the Reynolds number based on the Taylor microscale and already defined in Chapter 1 is $R_\lambda \approx 72{,}600$. This is huge compared with laboratory

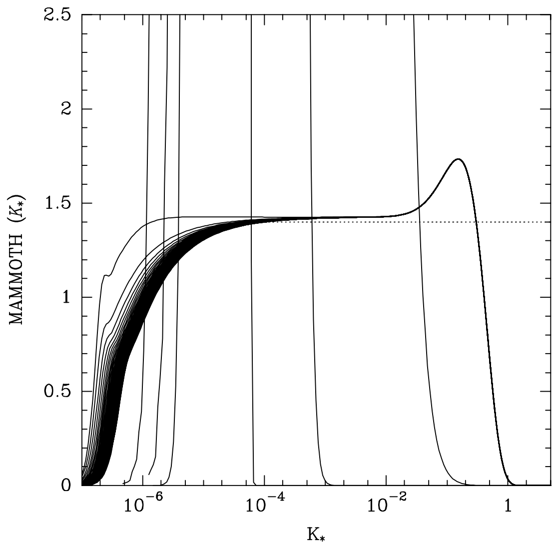

Figure 4.2. Kolmogorov-compensated kinetic-energy spectrum evolution (in dissipative units) corresponding to the EDQNM run of Figure 4.1.

or even environmental situations and might be encountered in astrophysics. Figure 4.2 shows for the run the Mammoth-shaped function $M(k_*, t)$ (introduced Chapter 1 on the r.h.s. of Eq. (1.47), with $k_* = k\eta$). The vertical lines correspond to spectra early at times. At later times, we get a perfect superposition of the curves at high wavenumbers, which indicates the validity of Kolmogorov similarity. At low wavenumbers, the dark area represents a decay of compensated spectra, which can be interpreted as the "Mammoth losing fat from the back."[4] At the end of the evolution there is a two-decade real compensated plateau at $C_K = 1.4$, and the spectral-bump size is one decade long. It is clear here that the limit of infinite Reynolds number, which would yield a Kolmogorov $k^{-5/3}$ spectrum extending to infinity, is just a mathematical view that cannot be reproduced in these calculations. However, Lesieur and Ossia [174] show that at such a high Reynolds number a limit curve is obtained for the skewness $s(t)$ defined by Eq. (1.33). The curve can be obtained from the following relation (see Orszag [225]):

$$s(t) = \left(\frac{135}{98}\right)^{1/2} D(t)^{-3/2} \int_0^{+\infty} k^2 T(k, t) dk, \tag{4.12}$$

[4] A former French minister for education used to say that he would remove the fat off the national education mammoth.

where $T(k, t)$ is the kinetic-energy transfer given here by the r.h.s. of Eq. (4.5). The time evolution from zero to infinity of this limit skewness displays first a rise to the maximum value of 1.132 attained at $t \approx 4.1$, then an abrupt drop to a plateau value of 0.547 reached at $t \approx 4.8$, and is conserved exactly above up to $t = 100$. This evolution is explained in Lesieur [170] as a transition between an initial inviscid skewness growth[5] to a skewness determined by a balance between vortex stretching and molecular dissipation terms in the r.h.s. of the enstrophy time-evolution equation. This yields a skewness constant with time if enstrophy and palinstrophy are assumed to be dominated by inertial and dissipative wavenumbers and scale on Kolmogorov dissipative units (Batchelor [18], Orszag [225]).

Let us return to the EDQNM Mammoth-shape compensated spectra. As stressed in Chapter 1, similar behaviors may be obtained from experimental data, with similar type of scalings, as reviewed for instance by Coantic and Lasserre [47], who have developed an analytical model to account satisfactorily for Reynolds-number changes in the experiments. The bump-shaped spectrum had already been observed in the EDQNM calculations of André and Lesieur [6]. The bump was interpreted as a "bottleneck effect" by Falkovich [89]. We will return to this point later. Concerning the departure from Kolmogorov similarity at small wavenumbers, we will see that the latter cannot be achieved with the $s = 8$ value taken initially; it is only for $s = 1$ that it may hold.

4.3 EDQNM plateau-peak model

As we did for the deterministic velocity and scalar fluctuations, we split the EDQNM kinetic-energy and scalar-variance transfers into interactions involving only modes smaller than k_C and those involving the others. The equations for the supergrid-scale velocity $\bar{E}(k, t)$ and scalar $\bar{E}_\rho(k, t)$ spectra are, respectively,

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) \bar{E}(k, t) = T_{<k_C}(k, t) + T_{>k_C}(k, t) \tag{4.13}$$

and

$$\left(\frac{\partial}{\partial t} + 2\kappa k^2\right) \bar{E}_\rho(k, t) = T^\rho_{<k_C}(k, t) + T^\rho_{>k_C}(k, t), \tag{4.14}$$

where $T_{<k_C}(k, t)$ and $T^\rho_{<k_C}(k, t)$ are the spectral transfers corresponding to resolved triads such that $k, p, q \leq k_C$ and $T_{>k_C}$ (resp. $T^\rho_{>k_C}$) transfer to modes

[5] We recall that Lesieur ([170], pp. 190–191) has shown for an initial-value problem in the framework of the Euler equation that, if $s(t)$ grows with time, or remains constant, or even decays slower than t^{-1}, then enstrophy will blow up in a finite time.

such that $k < k_C$, p and (or) $q > k_C$. We assume first that $k \ll k_C$ with both modes being larger than k_i, the kinetic-energy peak. Expansions in powers of the small parameter k/k_C yield to the lowest order

$$T_{>k_C}(k,t) = -2\nu_t^\infty \, k^2 \, \bar{E}(k,t), \tag{4.15}$$

$$\nu_t^\infty = \frac{1}{15}\int_{k_C}^{\infty} \theta_{0pp} \left[5E(p,t) + p\frac{\partial E(p,t)}{\partial p} \right] dp, \tag{4.16}$$

$$T_{>k_C}^{\rho}(k,t) = -2\kappa_t^\infty \, k^2 \, \bar{E}_T(k,t), \tag{4.17}$$

$$\kappa_t^\infty = \frac{2}{3}\int_{k_C}^{\infty} \theta_{0pp}^{\rho} \, E(p,t) \, dp. \tag{4.18}$$

Let us start by assuming a $k^{-5/3}$ inertial range at wavenumbers greater than k_C. We obtain

$$\nu_t^\infty = 0.441 \, C_K^{-3/2} \left[\frac{E(k_C)}{k_C} \right]^{1/2} \tag{4.19}$$

and

$$\kappa_t^\infty = \frac{\nu_t^\infty}{Pr^{(t)}} \tag{4.20}$$

with

$$Pr^{(t)} = 0.6. \tag{4.21}$$

Here, $E(k_C)$ is the kinetic-energy spectrum at the cutoff k_C. The 0.6 value for the Prandtl number is in fact the highest one permitted by the choice of two further adjustable constants arising in the EDQNM passive-scalar equation (see [170]). If we assume for instance a Kolmogorov constant of 1.4 in the energy cascade, the constant in front of Eq. (4.19) will be 0.267. When k is close to k_C, the numerical evaluation of the EDQNM transfers yields

$$T_{>k_C}(k,t) = -2\nu_t(k|k_C) \, k^2 \, \bar{E}(k,t) \tag{4.22}$$

and

$$T_{>k_C}^{\rho}(k,t) = -2\kappa_t(k|k_C) \, k^2 \, \bar{E}(k,t) \tag{4.23}$$

with

$$\nu_t(k|k_C) = K\left(\frac{k}{k_C}\right) \nu_t^\infty \tag{4.24}$$

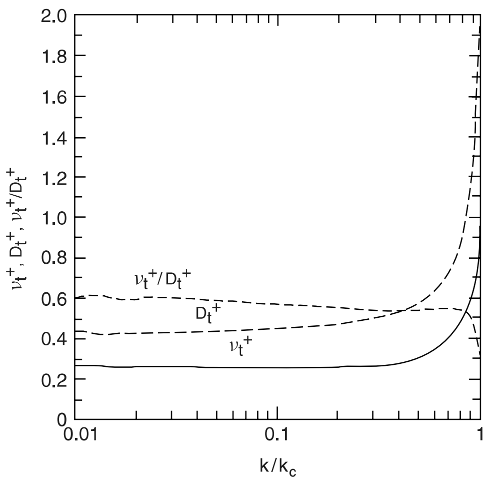

Figure 4.3. Eddy viscosity, eddy diffusivity, and turbulent Prandtl number in spectral space determined using the EDQNM theory. (From [42].)

and

$$\kappa_{\rm t}(k|k_{\rm C}) = C\left(\frac{k}{k_{\rm C}}\right)\kappa_{\rm t}^{\infty}, \tag{4.25}$$

where $\nu_{\rm t}^{\infty}$ and $\kappa_{\rm t}^{\infty}$ are the asymptotic values given by Eqs (4.19), (4.20), and (4.21), and $K(x)$ and $C(x)$ are nondimensional functions equal to 1 for $x = 0$. As shown also by Kraichnan's test-field model calculations [147], $K(x)$ has a plateau value at 1 up to $k/k_{\rm C} \approx 1/3$. Above, it displays a strong peak (cusp behavior). Let us mention that Kraichnan did not point out the scaling of the eddy viscosity against $[E(k_{\rm C})/k_{\rm C}]^{1/2}$, which turns out to be essential for LES purposes. Indeed, when the energy spectrum decreases rapidly at infinity (for instance during the initial stage of decay in isotropic turbulence), the eddy viscosity will be very low and inactive. However, we have $[E(k_{\rm C})/k_{\rm C}]^{1/2} \sim \epsilon^{1/3} k_{\rm C}^{-4/3}$ in an inertial-range expression. If we keep this inertial-range-type eddy viscosity before the establishment of the $k^{-5/3}$ range and evaluate ϵ as proportional to $E_c^{3/2} k_i$, it may substantially increase the eddy viscosity and work against the cascade development. We will explain in the following that the plateau-peak model may be generalized to spectra different from the Kolmogorov one at the cutoff (spectral-dynamic model).

It was shown in [42] that $C(x)$ behaves qualitatively as $K(x)$ (plateau at 1 and positive peak) and that the spectral turbulent Prandtl number $\nu_{\rm t}(k|k_{\rm C})/\kappa_{\rm t}(k|k_{\rm C})$ is approximately constant and thus equal to 0.6 as given by Eq. (4.21). These three quantities (eddy viscosity, eddy diffusivity, and turbulent Prandtl number) taken from [42] are shown in Figure 4.3 as a function of $k/k_{\rm C}$. In the figure, the eddy coefficients are normalized by $\sqrt{E(k_{\rm C})/k_{\rm C}}$ with $C_{\rm K} = 1.4$.

It is clear that the plateau part corresponds to the usual eddy-coefficients assumption when one goes back to physical space,[6] and thus the "peak" part goes beyond the scale-separation assumption inherent in the classical eddy-viscosity and diffusivity concepts. The peak is mostly due to semilocal interactions across k_C: Near the cutoff wavenumber, the main nonlinear interactions between the resolved and unresolved scales involve the smallest eddies of the former and the largest eddies of the latter (such that $p \ll k \sim q \sim k_C$). The peak also contains possible backscatter contributions (which are however very small if k_C lies in a Kolmogorov cascade) coming from subgrid modes larger than k_C. This point will be detailed in the following.

As shown in [43], the plateau-peak behavior of $K(x)$ can be approximately expresssed with the following analytical expression:

$$K(x) = 1 + 34.5\, e^{-3.03/x}. \tag{4.26}$$

We will see later another analytic expression of this spectral eddy viscosity in terms of hyperviscosities.

The plateau-peak model consists of using these eddy viscosities in the deterministic equations (4.1) and (4.3). One advantage of such a subgrid-scale modeling is that it is correct from an energy-transfer viewpoint. It is also able to deal with a continuous spectrum at the cutoff, which is a great asset with respect to the plain eddy-viscosity assumption in physical space. However, the assumption of real eddy coefficients is constraining and neglects the possible phase effects arising in the neighborhood of k_C.

4.3.1 Spectral-dynamic model

Another drawback of the plateau-peak model is that it is restricted to the case in which k_C lies within a $k^{-5/3}$ Kolmogorov cascade. Fortunately, this can be cured by introducing the spectral-dynamic model. We assume now that the kinetic-energy spectrum is $\propto k^{-m}$ for $k > k_C$ with m not necessarily equal to $5/3$. We modify the spectral eddy viscosity as

$$\nu_t(k|k_C) = 0.31\, C_K^{-3/2} \sqrt{3-m}\, \frac{5-m}{m+1} K\left(\frac{k}{k_C}\right) \left[\frac{E(k_C)}{k_C}\right]^{1/2} \tag{4.27}$$

for $m \leq 3$. This expression is exact for $k \ll k_C$ within the same nonlocal expansions of the EDQNM theory, as shown in Métais and Lesieur [205]. We retain the peak shape through $K(k/k_C)$ to be consistent with the Kolmogorov spectrum expression of the eddy viscosity. For $m > 3$, the scaling is no longer

[6] There is, however, a slight difference at this level because going back to physical space will give ν_t^∞ multiplied by the filtered-velocity Laplacian, whereas, in the physical-space formalism, the eddy viscosity is under a divergence operator in Eq. (3.19).

valid, and the eddy viscosity will be set equal to zero. Indeed, we are very close to a DNS for such spectra. In the spectral-dynamic model, the exponent m is determined through the LES with the aid of least-squares fits of the kinetic-energy spectrum close to the cutoff. We may also check that the turbulent Prandtl number is given by

$$Pr^{\mathrm{t}} = 0.18\,(5 - m) \tag{4.28}$$

(see Métais and Lesieur [205] and Lesieur [170], p. 386). This value does not depend of the Kolmogorov and model constants. Being able to use a variable turbulent Prandtl number is a great advantage in LES of heated or variable-density flows. This possibility exists also for the dynamic models in physical space such as the dynamic Smagorinsky model presented in Chapter 3.

4.3.2 Spectral random backscatter

There are many discussions on LES related to the concept of random backscatter, one aspect of which in physical space is the negativeness of the eddy viscosity in local regions. We give here some elements of this discussion in Fourier space. We return to the EDQNM kinetic-energy transfer $T(k,t)$ in three-dimensional isotropic turbulence given by the r.h.s. of Eq. (4.5). Such a transfer may be rewritten by a symmetrization with respect to p and q in the integrand: In the first term $a(k,p,q) = (1/2)[b(k,p,q) + b(k,q,p)]$ appears, which may be shown to be positive (see Orszag [225] and Lesieur [170]). The second term is proportional to $k^2 E(k,t)$. This ensures the realizability (positiveness of the kinetic-energy spectrum) of the closure. We consider now some arbitrary cutoff wavenumber k_{C}, which is not necessarily in the middle of an inertial range. The subgrid kinetic-energy transfer across k_{C} is then

$$T_{\mathrm{sg}}(k) = A_{\mathrm{BS}} - B_{\mathrm{D}}, \tag{4.29}$$

where A_{BS}, the backscatter term, is given by

$$\begin{aligned} A_{\mathrm{BS}} = k^4 \int_{k_{\mathrm{C}}}^{\infty} dp \int_{k/2p}^{1} \frac{1-z^2}{q^2} \\ \times \left[1 + \frac{p^2}{q^2} + \left(\frac{k}{q} - 2\frac{p}{q}z\right)^2\right] \theta_{kpq} E(p) E(q)\, dz. \end{aligned} \tag{4.30}$$

This term is obviously positive. The second term can be written as

$$\begin{aligned} B_{\mathrm{D}} = k^2 E(k) \int_{k_{\mathrm{C}}}^{\infty} dp \int_{k/2p}^{1} \theta_{kpq}(1-z^2) \\ \times \left[\left(\frac{p^2}{q^2} - \frac{pz}{k}\right)\frac{p^2}{q^2}E(q) + \left(1 - \frac{p^2}{q^2} + \frac{p^3 z}{kq^2}\right)E(p)\right] dz. \end{aligned} \tag{4.31}$$

These expressions can be simplified if $k \ll k_C$ (which implies that p and q are of the same order). Then,

$$A_{BS} = \frac{14}{15} k^4 \int_{k_C}^{\infty} \theta_{0pp} \frac{E(p)^2}{p^2} dp, \tag{4.32}$$

$$B_D = 2\nu_t^{\infty} k^2 E(k), \tag{4.33}$$

where the eddy viscosity ν_t^{∞} has been given in Eq. (4.16). If k and k_C both lie in the inertial range (with $k \ll k_C$), the k^4 backscatter is of the order of $k^4 \theta_{0,k_C,k_C} k_C^{-1} E(k_C)^2$, and the eddy-viscosity contribution is of the order of $k^2 E(k) \theta_{0,k_C,k_C} k_C E(k_C)$. Hence, in this case

$$\frac{A_{BS}}{B_D} \sim \left(\frac{k}{k_C}\right)^2 \frac{E(k_C)}{E(k)}, \tag{4.34}$$

which is very small for any decreasing kinetic-energy spectrum. This justifies the fact that the plateau part of the spectral eddy viscosity considered here does not include any k^4 backscatter contribution. However, backscatter is important when k is close to k_C, but the coefficient in front of k^4 is not a simple function of k; moreover, it is difficult to tell the exact k dependence of the backscatter in this case or of the eddy-viscosity term B_D. What is certain is that the plateau-peak eddy viscosity does properly include the backscatter at the level of correct kinetic-energy exchanges.

In fact, the k^4 backscatter transfer plays an important role in the infrared part of the spectrum ($k \to 0$). We assume that $k_C = k_i$ corresponds to the peak of the spectrum and again $k \ll k_C$. Now the backscatter given by Eq. (4.32) dominates the local transfers. It injects energy in very large scales through resonant interaction of two energetic modes, and it is responsible for the immediate emergence of an infrared k^4 spectrum in isotropic decaying turbulence when energy is injected initially at a peak at k_i. This point, predicted by two-point closures (see Lesieur and Schertzer [164] and Lesieur [170]), was first checked in LES of isotropic turbulence by Lesieur and Rogallo ([165]; see Figure 4.4) using the plateau-peak model, and we will confirm it with LES using the spectral-dynamic model.

In forced stationary turbulence obtained when a random statistically stationary forcing is applied on a narrow spectral band around k_i, the net infrared transfer is given by the combination of the backscatter and the eddy-viscous drain. It should vanish because the energy spectrum is time invariant. There is then a balance between the k^4 backscatter and the $k^2 E(k)$ drain, which yields a k^2 equipartition spectrum.

We stress finally that in a turbulent mixing-layer calculation, Leith [162] used a k^4 random backscatter forcing as a way to inject energy into the large scales.

4.4 Return to the bump

We have already mentioned for decaying isotropic turbulence the "bump" existing at the edge of the Kolmogorov $k^{-5/3}$ inertial range before the dissipative range. In fact, forced EDQNM calculations with a narrow forcing at k_i do show the persistence of the bump (Mestayer et al. [203], Lesieur and Ossia [174]). In the calculations of Mestayer et al. [203], the bump did disappear with the removal of nonlocal triads (k, p, q) of the type $k < ap$ (with $a \approx 2^{1/F} - 1 \approx 0.2$ when taking $F = 4$). These elongated nonlocal interactions correspond to an energy flux given by (see [170], p. 233, for details)

$$\Pi_{\mathrm{EL}}(k,t) = \frac{2}{15}\int_0^k k'^2 E(k')dk' \int_{\sup(k,k'/a)}^{\infty} \theta_{k'pp}\,[5E(p) + p\frac{\partial E}{\partial p}]dp$$
$$- \frac{14}{15}\int_0^k k'^4 dk' \int_{\sup(k,k'/a)}^{\infty} \theta_{k'pp}\,\frac{E(p)^2}{p^2}dp. \tag{4.35}$$

The first term in Eq. (4.35) is of the "eddy-viscous type"; the second is of the "backscatter type," but the latter may be checked to be negligible in the energy cascade, as already stressed. No real explanation for the bump disappearance is given in Mestayer et al. [203], who just note that "the bumps appear to result mainly from a lack of erosion of the spectra by elongated non-local interactions when approaching the viscous cutoff." Falkovich [89] interpreted the bump as a "bottleneck phenomenon . . . where a viscous suppression of small scale modes removes some triads from nonlinear interactions . . . which leads to a pileup of the energy in the inertial interval of scales." In fact, this may be made more quantitative by looking back at the evaluation of the elongated nonlocal flux given by Eq. (4.35) carried out in [203]. It is positive and approximately constant in the inertial range. We will assume that it would remain constant in a $k^{-5/3}$ range extending to infinity. However, because of its structure in terms of integrals to infinity upon the energy spectrum, the elongated flux should start to decrease rapidly when feeling nonlocally the dissipative range, which is much further upstream. If we assume that the local and other nonlocal fluxes are not yet affected by dissipation, and hence are still constant, the global flux will be decreased, implying a positive kinetic-energy transfer, resulting in the bump.

4.5 Other types of spectral eddy viscosities

4.5.1 Heisenberg's eddy viscosity

In fact, the concept of a wavenumber-dependant eddy viscosity may already be found in Heisenberg ([120], see also Mc Comb [200] for details). Heisenberg introduced this eddy viscosity to model the evolution of the kinetic-energy spectrum. Within this model, and as recalled by Schumann [260], the derivative of the eddy viscosity with respect to k is proportional to $-\sqrt{E(k)/k^3}$. If we assume some power-law dependence for the kinetic-energy spectrum, Heisenberg's eddy viscosity will indeed scale as $\sqrt{E(k)/k}$. This is a type of local spectral eddy viscosity, which is less rich than the nonlocal plateau-peak formulation. It was used by Aubry et al. [10] to model equivalent subgrid scales in the dynamical system describing the evolution of a turbulent boundary layer within a proper orthogonal decomposition (POD) approach. We recall that in the POD (see Holmes et al. [126] for a review), the velocity vector is projected on the eigenvectors of the Reynolds-stress tensor. In this context, ejection or sweep events occurring in the boundary layer appeared as particular events in a chaotic dynamical system.

4.5.2 RNG analysis

Another approach, the renormalization group (RNG) method, originally developed by Forster et al. [97] and Fournier [99] for isotropic turbulence, has been applied by Yakhot and Orszag [293] and McComb [200] to LES with an eddy viscosity proportional to $\sqrt{E(k_C)/k_C}$. Let us recall briefly the RNG formalism in Fournier's work. In classical RNG analysis applied to the physics of critical phenomena, the dimension d of space is considered as a variable parameter. In general, the problem can be solved analytically for the dimension $d = 4$. Then the solution for $d = 4 - \epsilon$ is obtained from this solution through expansions in powers of the parameter ϵ, which is assumed to be small. The solution for $d = 3$ is recovered by making $\epsilon = 1$. Although slightly awkward, the procedure works remarkably well for various problems such as spin dynamics in ferromagnetic systems. Forster et al. [97] adapted the method to the Navier–Stokes equation with a varying dimension of space. In contrast, Fournier works with a fixed dimension of space (three), and he considers a kinetic-energy forcing term proportional to k^{-r} with a varying exponent r. One supposes at a given time that energy is distributed on a wavenumber interval $[0, \Lambda]$. Let $\delta\Lambda \ll \Lambda$, and let $\Lambda - \delta\Lambda$ be a sort of cutoff wavenumber with $\delta\Lambda/\Lambda$ fixed. The velocities corresponding to modes in the shell $[\Lambda - \delta\Lambda, \Lambda]$ are solved, through Feynman diagrammatic perturbation techniques involving Green-function operators, in terms of modes smaller than $\Lambda - \delta\Lambda$. Statistical independence between the "subgrid" and "supergrid" modes is further assumed. A new Navier–Stokes equation with renormalized eddy viscosity and forcing, involving the wavenumber interval $[0, \Lambda - \delta\Lambda]$, is written. One has thus eliminated ("decimated") the shell $[\Lambda - \delta\Lambda, \Lambda]$. As stressed by Lesieur ([170] p. 253), other terms, called "nonpertinent," still arise at this level, but these will vanish after an infinite number of decimations. Then the operation is iterated an infinite number of times to let the cutoff Λ go to zero. The small parameter here is $\epsilon = 3 + r$. For $\epsilon > 0$, one obtains (Fournier and Frisch [100], Lesieur [170], p. 255) an eddy viscosity proportional to $\sqrt{E(\Lambda)/\Lambda}$. However, to obtain a Kolmogorov $k^{-5/3}$ kinetic-energy spectrum requires $r = 1$, so that the "small" parameter is now 4, which is excessive and cannot guarantee the convergence of the expansions. Furthermore, this expression of the renormalized eddy viscositiy is valid only for $\Lambda \to 0$, whereas it is used for LES purposes with a finite cutoff for which the nonpertinent terms cannot be neglected. Finally, there is no general consensus about the determination of the numerical constant arising in the eddy viscosity.

These results indicate that the plateau-peak model has the richest dynamics of all the Heisenberg-type $\sim\sqrt{E(k_C)/k_C}$ eddy viscosities.

4.6 Anterior spectral LES of isotropic turbulence

The plateau-peak eddy viscosity was applied by Chollet and Lesieur [41] to the first spectral LES of three-dimensional isotropic turbulence (a pseudo-spectral method with a resolution of 32^3 Fourier collocation modes). They studied decaying turbulence; there is no molecular viscosity,[7] and the initial energy spectrum decreases rapidly at infinity. During the first stage the kinetic energy is transferred toward k_C accompanied by a growth of the resolved enstrophy $D(t)$. At about four initial large-eddy turnover times $D(0)^{-1/2}$, the enstrophy reaches a maximum and decreases, whereas the kinetic-energy spectrum decays self-similarly with an approximate $k^{-5/3}$ slope.

Large-eddy simulations of a passive scalar at the same resolution were performed by these authors in 1982 with qualitatively the same results and the formation of a $k^{-5/3}$ Corrsin–Oboukhov inertial-convective scalar spectrum. These results are presented in Lesieur ([170], p. 389). However, using 32^3 collocation points gives extremely low resolution and is totally unable to capture the fine features of isotropic turbulence. We show in Figure 4.4 an analogous LES at a resolution of 128^3 Fourier modes[8] carried out by Lesieur and Rogallo [165]. The initial velocity and scalar spectra are proportional with a Gaussian ultraviolet behavior and a k^8 infrared spectrum. It can be checked that Kolmogorov and Corrsin–Oboukhov $k^{-5/3}$ cascades are established. Afterward, the kinetic-energy spectrum decays self-similarly with a spectral slope between $-5/3$ and -2. The scalar spectrum seems to have a

[7] It is a nice behavior of these large-eddy simulations to allow for "Euler LES" without any numerical energy diffusion. The question posed is of course of the relevance of the solutions found with respect to real solutions of Euler equations or of the Navier–Stokes equation in the limit of zero viscosity.

[8] Such a simulation was not dealiased, but it is now well recognized that, in contrast to DNS, aliasing effects may be important in spectral LES and should be eliminated.

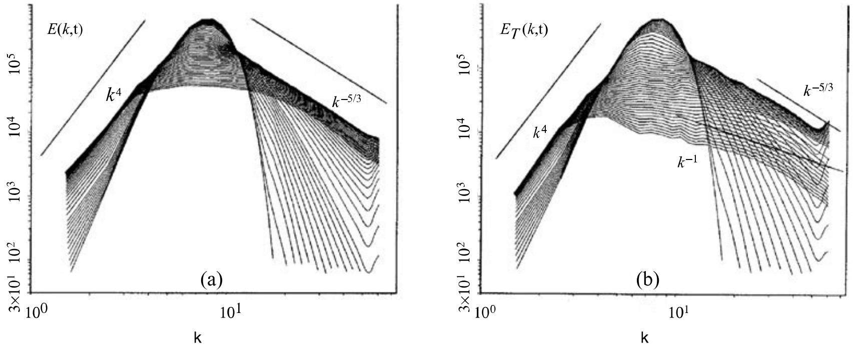

Figure 4.4. Three-dimensional isotropic decaying turbulence showing decay of kinetic-energy (a) and passive-scalar (b) spectra in the LES of Lesieur and Rogallo [165] using the plateau-peak eddy viscosity.

very short inertial-convective range close to the cutoff and a very wide range shallower than k^{-1} in the large scales. Here, the scalar decays in time much faster than the temperature. This anomalous range was explained by Métais and Lesieur [205] as due to the quasi-two-dimensional character of the scalar diffusion in the large scales, leading to large-scale intermittency of the scalar. More precisely, the scalar diffusion seems to be dominated by the effect of the coherent vortices already considered in Chapter 2. More details on this anomalous k^{-1} range may be found in Lesieur ([170], p. 211).

4.6.1 Double filtering in Fourier space

These spectral LESs of decaying isotropic turbulence and associated scalar mixing, together with those of Métais and Lesieur [205], have been used to compute directly the spectral eddy viscosity and diffusivity. The method is the same as that employed by Domaradzki et al. [70] for a DNS: One defines a fictitious cutoff wavenumber $k'_C = k_C/2$ across which the kinetic-energy transfer T and scalar transfer T^ρ are evaluated. Because we deal with a LES, the latter corresponds to triadic interactions such that $k < k'_C$, p and (or) $q > k'_C$ and $p, q < k_C$. These are termed $T_{>k'_C}^{<k_C}(k,t)$ and $T_{>k'_C}^{T<k_C}(k,t)$. They correspond to resolved transfers and satisfy energetic equalities of the type

$$T_{>k'_C}^{<k_C}(k,t) = T_{>k'_C}(k,t) - T_{>k_C}(k,t), \tag{4.36}$$

where $T_{>k'_C}$ and $T_{>k_C}$ are the total kinetic energy transfers across k'_C and k_C. It is important to note that Eq. (4.36) is the exact energetic equivalent in spectral space of Germano's identity if one works in Fourier space with sharp filters. A similar relation holds for $T_{k'_C}^{T<k_C}$. Dividing these equations by $-2k^2\,E(k,t)$ and $-2k^2\,E_\rho(k,t)$ gives the resolved spectral eddy-viscosity and diffusivity.

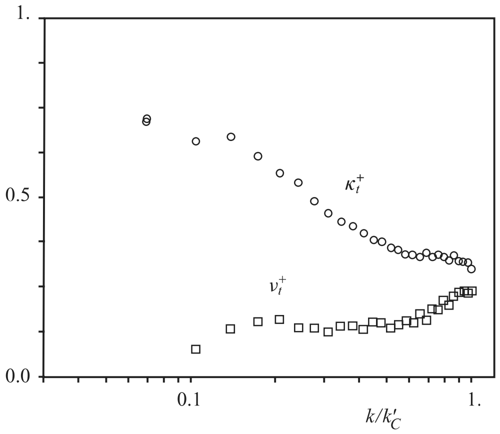

Figure 4.5. Resolved eddy viscosity and diffusivity evaluated through a double filtering in LES of isotropic decaying turbulence. (From Métais and Lesieur [205].)

Figure 4.5 shows these functions normalized by $[E(k'_C)/k'_C]^{1/2}$. Similar results have been found in Lesieur and Rogallo [165]. The figure demonstrates that the plateau-peak behavior does exist for the eddy viscosity but is questionable for the eddy diffusivity. This anomaly is obviously related to the anomalous scalar k^{-1} range previously mentioned. It was stressed by Lesieur ([170], p. 392) that this anomalous scalar range still exists in a DNS of decaying isotropic turbulence: In this case, the double filtering yields a plateau-peak eddy viscosity with a plateau value of approximately zero, as was discovered by Domaradzki et al. [70]. The eddy diffusivity, in contrast, still behaves as in the LES. In fact, Métais and Lesieur [205] have checked that the anomaly disappears when the temperature is no longer passive and is coupled with the velocity within the framework of the Boussinesq approximation (stable stratification). It is possible that the same holds for compressible turbulence, which would legitimize the use of the plateau-peak eddy diffusivity in this case.

4.7 EDQNM infrared backscatter and self-similarity

We return now to the EDQNM analysis of Lesieur and Ossia [174] and show that it is only at $s = 1$ that the kinetic-energy spectrum may have a global self-similarity at entire scales from the energy-containing to the dissipative ones. The derivation is borrowed from Lesieur and Schertzer [164], who applied it to the EDQNM spectral equation. We present a generalization that does not require use of closure. The first point is to remark that such a global self-similarity necessarily implies that the integral and dissipative scales l and l_D are proportional, with their ratio being time independent. If a Karman–Howarth self-similarity is assumed, the kinetic-energy and transfer spectra are, respectively,

$$E(k,t) = v^2 l \, F(kl), \quad T(k,t) = v^3 \, T_1(kl), \tag{4.37}$$

where the functions F and T_1 are nondimensional. We assume in fact that a regime is reached such that all the quantities have an algebraic time dependance, and thus we can write

$$E(k,t) = t^n \, G(k'), \quad T(k,t) = t^{3(n-m)/2} \, T'(k') \tag{4.38}$$

with $k' = kt^m$, $v^2 \propto t^{n-m}$, and $l \propto t^m$. Notice here that G and T' are dimensional functions of the dimensional argument k'. Substituting these expressions into the kinetic-energy spectrum evolution equation

$$\frac{\partial E}{\partial t} + 2\nu k^2 E = T(k,t), \tag{4.39}$$

we obtain (after division by $t^{3(n-m)/2}$)

$$\left[nG + mk' \frac{dG}{dk'} \right] t^{(3m-n-2)/2} + 2\nu k'^2 G(k') t^{-(m+n)/2} = T'(k'). \tag{4.40}$$

In this equation, all the terms have to be time independent. We do have $3m - n - 2 = 0$ (a condition that we could have obtained by writing $\epsilon \sim v^3/l$) and $m + n = 0$, which finally implies that $m = 1/2$, $n = -1/2$, and α_E (such that $v^2 \propto t^{-\alpha_E}$) is equal to $m - n = 1$. It is a well-known result that such a global self-similarity, when applied to the infrared spectrum, implies a further condition. Indeed, relation (4.38) gives for an infrared kinetic-energy spectrum $\propto t^{\gamma_s} k^s$ (for which viscosity has a negligible effect if small enough)

$$\gamma_s = n + ms \tag{4.41}$$

and $s = 1 + 2\gamma_s$. We know that (in the EDQNM framework where a k^4 backscatter is assumed), γ_s is zero except for $s = 4$, where it is equal to 0.16 (see Lesieur and Schertzer [164] and Lesieur [170]), and the only possibility is thus $s = 1$. Hence, the large-scale Reynolds number R_l should be constant with time as well as $R_\lambda \sim \sqrt{R_l}$.

The question of permissible values for s is a controversial one. There are arguments in favor of $s = 4$ (see the review of Davidson [62]) and others in favor of $s = 2$ (Saffman [247]). Taking even values of s is compulsory if certain regularity conditions are fulfilled for the velocity-correlation tensor between two points when the distance goes to infinity. Mathematically, we may take initially odd values of s (such as 1 or 3) and even noninteger ones, as was proposed by Eyink and Thomson [88]. Working on the basis of an analogy with Burgers turbulence studied by Gurbatov et al. [116] with DNS, Eyink and Thomson propose the existence of a crossover dimension $s \approx 3.45$, above which a k^4 backscatter should appear. The crossover value is obtained